Arbitrary Pattern Formation on a Continuous Circle by Oblivious Robot Swarm[⋆]

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Abstract. In the field of distributed system, Arbitrary Pattern Formation (APF) problem is an extensively studied problem. The purpose of APF is to design an algorithm to move a swarm of robots to a particular position on an environment (discrete or continuous) such that the swarm can form a specific but arbitrary pattern given previously to every robot as an input. In this paper the APF problem is discussed on continuous circle by a swarm of homogeneous, anonymous, oblivious, silent mobile robots with no chirality. The algorithm discussed here, can solve APF problem deterministically within $\mathcal{O}(n)$ epochs without collision under Semi Synchronous scheduler with n (where n is odd) mobile robots if the initial configuration is rotationally asymmetric. For any even number of robots (grater than three) the algorithm can be modified to a probabilistic one to solve APF. Also if the initial configuration is rotationally symmetric then APF problem can't be solved by any deterministic algorithm.

Keywords: APF · Continuous circle · Oblivious · mobile robots · distributed algorithm.

1 Introduction

Applications of distributed systems and their relevant problems have gained substantial importance in the last two decades. Unlike a centralized system, using a swarm of inexpensive, simple robots to do a task is more cost-effective, robust, and scalable. These swarms of robots have many applications like rescue operations, military operations, search and surveillance, rescue operations, disaster management, cleaning of large surfaces, and so on.

Researchers are interested in studies about using swarm robots with minimum capabilities to do some specific tasks like GATHERING, ARBITRARY PATTERN FORMATION, DISPERSION, EXPLORATION, SCATTERING etc. The robots are *autonomous* (have no centralized controller), *anonymous* (have no IDs), and *homogeneous* (have same capabilities and execute same algorithm). Depending on the capabilities of robots there are four types of robot models: $\mathcal{OBLOT}$,

$\mathcal{FSTA}$, $\mathcal{FCOM}$, $\mathcal{LUMI}$. In the $\mathcal{OBLOT}$ model robots do not have any persistent memory of their previous round i.e. *oblivious* and they can't communicate with each other i.e. *silent*. In the $\mathcal{FSTA}$ model robots are not oblivious but silent. In the $\mathcal{FCOM}$ model robots are oblivious but not silent. In the $\mathcal{LUMI}$ model robots are neither oblivious nor silent.

Each robot executes a Look-Compute-Move (LCM) cycle after activation. In Look phase robot takes a snapshot of its surroundings and collects the required information. Then the robot calculates the target using that information in the Compute phase and moves to the destination in Move phase. A scheduler is the controller of the activation of robots. There are three types of schedulers: *Fully synchronous* (FSync) scheduler, *Semi synchronous* (SSync) scheduler, *Asynchronous* (ASync) scheduler. In FSync scheduler, time is divided into global rounds of the same duration and each robot is activated in every round and executes the LCM cycle. In SSync scheduler, here also time is divided into global rounds of the same duration like FSync, but all robots may not be activated at the beginning of each round. In ASync scheduler robots are activated independently and the LCM cycle is not synchronized here.

The problem considered here is the Arbitrary Pattern Formation (APF) problem in which a swarm of robots is deployed in an environment (discrete or continuous domain). APF problem aims to design an algorithm such that robots move to a particular position and form a specific but arbitrary pattern which is already given to every robot as input. There is a vast literature of APF in both discrete and continuous domains ([18,19,14,11,4,5,7,2,3,1,8,16,6]). Most of the works of APF in continuous domain considered on the Euclidean plane. There are other sorts of environments which are included continuous domain, e.g. any closed curve embedded on the plane where robots can only move on that curve. In real life, such environments also exist and are hugely applicable in different scenarios such as roads, railway tracks, tunnels, waterways, etc. Another example of this kind of environment can be a circle of fixed radius embedded on the plane. Studying this problem is interesting because the solution for this problem can be extended to all other closed curves. Thus in this paper, we have considered the problem of Arbitrary Pattern Formation (APF) on a circle.

2   Related Works and Our Contribution

2.1   Related work

In swarm robotics, Arbitrary Pattern Formation (APF) problem is a hugely studied problem. This problem was first introduced by Suzuki and Yamashita in [18] on the Euclidean plane. Later they characterized the geometric patterns formable by oblivious and anonymous robots in [19] for fully synchronous and asynchronous scheduler. After that this problem has been considered on different environments on continuous and discrete domains ([14,11,4,5,7,2,3,1,8,16,6]).

In the continuous domain, most of the works which consider arbitrary pattern formation are done on the Euclidean plane under different settings. In [14]

Flocchini discussed the solvability of the pattern formation problem by considering oblivious robots with fully asynchronous schedulers. They showed that if the robots have no common agreement with their environment, they are unable to form an arbitrary pattern. Moreover, if the robots have one axis agreement, then any odd number of robots can form an arbitrary pattern, whereas the even number can't. Further, if the robots have both axis agreement, then any set of robots can form any pattern. They proved that it is possible to elect a leader if it is possible to form any pattern for $n \geq 3$ robots. The converse of this result is proved and a relationship between leader election and arbitrary pattern formation for oblivious robots in the asynchronous scheduler is studied in [11]. The authors showed that for $n \geq 4$ robots with chirality (respectively for $n \geq 5$ without chirality), the Arbitrary pattern formation problem is solvable if and only if leader election is possible. In [4] the authors proposed a probabilistic pattern formation algorithm for oblivious robots under asynchronous scheduler without chirality. Their protocol is the combination of two phases, one is a probabilistic leader election phase and another is a deterministic pattern formation phase. Later in [5] they proposed a new geometric invariant that exists in any configuration with four oblivious anonymous mobile robots to solve arbitrary pattern formation problems with or without the common chirality assumption. In [7] authors studied Embedded Pattern Formation without chirality with oblivious robots. They characterized when the problem can be solved by a deterministic algorithm and when it is unsolvable. In [2] authors studied the APF problem for the robots whose movement can be inaccurate and the formed pattern is very close to the given pattern. In [3] the authors provided a deterministic algorithm in the Euclidean plane with asynchronous opaque robots.

Note that all the works in arbitrary pattern formation considering continuous domain have been done only for the Euclidean plane where the robots can arbitrarily move from one point to another via infinitely many paths. But there are some environments in continuous domain in which the movements of robots from one point to another are restricted to a finite number of possible paths. A continuous circle of fixed radius is one such environment. To the best of our knowledge, there are some works ([17,13,15,9,10,12]) which considered continuous circle as their corresponding environment. The problem of patrolling, gathering, and rendezvous are the main focus of these works. But none of them considered the problem of APF on the continuous circle.

2.2   Our Contribution

In this work, our aim is to solve the problem of Arbitrary Pattern Formation (APF) on a continuous circle by oblivious and silent mobile robots with full visibility under semi-synchronous scheduler. To the best of our knowledge, the APF problem has not yet been considered on a continuous circle. So in this paper, we have considered this problem for the first time. Here the robots do not agree with a particular direction i.e. robots have no chirality. The movements of robots are restricted only in two directions, clockwise and anti-clockwise from any point. So, avoiding collision in a circle is more difficult than avoiding collision on a plane.

The robot model considered here is the weakest $\mathcal{OBLOT}$ model. In this problem, there is no particular landmark or door from which the robots enter. Here we discussed that, if the initial configuration is rotationally asymmetric then, for any odd number of robots the proposed deterministic algorithm APF_CIRCLE can solve the problem of Arbitrary Pattern Formation under Semi Synchronous (SSYNC) scheduler within $O(n)$ epochs avoiding collision, where n is the number of robots in the swarm. For any even number of robots, which is greater than three, the algorithm can be modified to a probabilistic one that solves APF on a continuous circle within $O(n)$ epochs. We have also shown that the APF problem can't be solved deterministically if the initial configuration is rotationally symmetric.

3   Model and Problem Definition

3.1   Problem Definition

Let $\mathcal{CIR}$ be a continuous circle of fixed radius. Let n robots resides on the perimeter of $\mathcal{CIR}$ such a way that the configuration formed by them is rotationally asymmetric. The robots can move freely on the circle. A sequence of angular distances $\beta_0, \beta_1, \beta_2 \ldots, \beta_{n-1}$ is given to the robots such that the sum of the angles of the sequence is equal to 2π. The problem is to design a distributed algorithm for the robots so that by finite execution of the algorithm, the robots move in such locations on $\mathcal{CIR}$, such that the final configuration has the following property:

✧ There exist a robot, say r_0 and a direction $\mathcal{D} \in \{$ clockwise, anticlockwise$\}$ such that the angular distance between the i-th and $(i+1)$-th robot in the direction $\mathcal{D}$ (denoted as r_i and r_{i+1} respectively) is β_i where all the indices are considered under modulo n.

3.2   Model

Robot Model: All robots are placed on the perimeter of a circle, say $\mathcal{CIR}$. Here the robots can move only on the perimeter of the circle. Robots have no particular orientation (i.e., no agreement on clockwise or anticlockwise direction). Robots have full visibility of the circle. The initial configuration is rotationally asymmetric. The movements of the robots are rigid i.e. robots always moves to its destination in a particular round. Robots have following properties-

Autonomous: Robots don't have any centralised controller.
Anonymous: Robots have no IDs.
Homogeneous: All robots have same capabilities and execute the same algorithm.
Oblivious: Robots have no persistent memory.
Silent: Robots have no means of communications.
Visibility: Robots have full visibility of the circle i.e., robots can see all other robots on the circle.

LCM cycle: Each robot executes a cycle of Look-Compute-Move(LCM) phases upon activation.

> *LOOK:* In look phase a robot takes a snapshot of its surroundings and gets the location of other robots on the circle according to its own coordinate .
> *COMPUTE:* Robot determines target location using the input from the snapshot of the look phase by executing the provided algorithm.
> *MOVE:* In move phase robot moves to the destination point calculated in the compute phase.

Scheduler Model: The activation of robots are controlled by an entity called scheduler. Depending on the activation timing there are three types of schedulers:

> *Fully synchronous* (FSync): In fully synchronous scheduler time is divided into rounds of equal length and all robots are activated at the beginning of every round and performs the LCM cycle synchronously.

> *Semi synchronous* (SSync): Similar to fully synchronous scheduler here also time is divided into rounds of equal length. But all robots might not get activated at the beginning of a particular round. In a particular round the activated robots perform the LCM cycle synchronously. Note that semi synchronous scheduler is more generalised than fully synchronous scheduler.

> *Asynchronous* (ASync): In asynchronous scheduler time is not divided into rounds like fully synchronous and semi synchronous scheduler. Robots are activated independently. In a particular moment of time some robots may be in Look phase, some in Compute phase, some in Move phase or some may be idle. Asynchronous scheduler is the most general among all the schedulers. In this paper, Semi Synchronous (SSync) scheduler has been considered to solve the problem.

4 Primineries

In this section, we first justify the reason for assuming the initial configuration is rotationally asymmetric. Then, we define some terminologies here in this section that will be needed to describe the algorithm provided in the next section.

Proposition 1. *There is no deterministic distributed algorithm that solves arbitrary pattern formation problem on a continuous circle if the initial configuration is rotationally symmetric.*

Proof. Let $\mathcal{C}(0)$ be the initial configuration which is rotationally symmetric. Let $\mathcal{C}(0)$ has a $k-$fold symmetry i.e., for a rotation of $\frac{2\pi}{k}$ along the center, the configuration remains the same. By proposition 2.4 in [17] if the scheduler is fully synchronous then for any algorithm $\mathcal{A}$, the new configuration will have a

$k'-$fold symmetry after one execution of $\mathcal{A}$, where $k' \geq k$. Thus for any finite execution of $\mathcal{A}$, the configuration will always have a k_1-fold symmetry where $k_1 \geq k$. So if the target pattern is asymmetric it can not be formed by the robot swarm by finite execution of $\mathcal{A}$. Hence the result.   □

Definition 1 $((A, B)_\mathcal{D})$. *Let A and B be two points on the circle $\mathcal{CIR}$. Let $\mathcal{D}$ be a direction either clockwise or anticlockwise. Then $(A, B)_\mathcal{D}$ denotes the angular distance from point A to point B in the direction $\mathcal{D}$.*

Definition 2 (Set of Angle Sequences of a robot r).
Let, $\mathcal{R} = \{r_0, r_1, r_2, \ldots, r_{n-1}\}$ be the set of robots placed consecutively in a fixed direction, say $\mathcal{D}$ (either clockwise or anticlockwise) on the circle $\mathcal{CIR}$. Let $\theta_{i \ (\mathrm{mod}\ n)}$ be the angular distance from the location of robot $r_{i \ (\mathrm{mod}\ n)}$ to the location of robot $r_{i+1 \ (\mathrm{mod}\ n)}$ in the direction $\mathcal{D}$. Then the set of angle sequences of the robot $r = r_0$, denoted as $\mathcal{AS}(r)$, is the set $\{\mathcal{AS}_D(r), \mathcal{AS}_{D'}(r)\}$ where, $\mathcal{D}'$ is the opposite direction of $\mathcal{D}$ and $\mathcal{AS}_D(r) = (\theta_0, \theta_1, \ldots, \theta_{n-1})$, $\mathcal{AS}_{D'}(r) = (\theta_{n-1}, \theta_{n-2}, \ldots, \theta_0)$ are two angle sequences in the direction $\mathcal{D}$ and $\mathcal{D}'$ respectively.

Since the initial configuration is rotationally asymmetric then, for a fixed particular orientation (either clockwise or anti-clockwise) all the robots have different angle sequences (??). So, note that, for two robots, say r_1 and r_2 if $\mathcal{AS}_{\mathcal{D}_1}(r_1) \in \mathcal{AS}(r_1)$ is equal to $\mathcal{AS}_{\mathcal{D}_2}(r_2) \in \mathcal{AS}(r_2)$ then, $\mathcal{D}_1$ must be equals to $\mathcal{D}_2'$

Definition 3 (Nominee). *A robot r_l is considered as the nominee if $min(\cup_{r \in \mathcal{R}} \mathcal{AS}(r)) \in \mathcal{AS}(r_l)$*
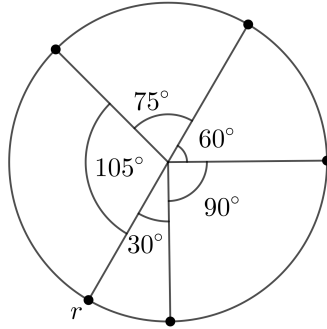

Fig. 1: Here, $\mathcal{AS}(r) = \{(30°90°60°75°105°), (105°75°60°90°30°)\}$ is the set of angle sequences for the robot r. Note that $\mathcal{AS}(r)$ contains the minimum angle sequence $(30°90°60°75°105°)$ so, r is a nominee. This configuration is also a single nominee configuration.

Proposition 2. *In the initial configuration, there are at least one and at most two nominees.*

Proof. Since the initial configuration is rotationally asymmetric, by a result stated in [17] it can be said that all the robots have distinct angle sequences in a particular direction. Thus n robots have n distinct angle sequences in a particular direction. Similarly in opposite direction, there exists n distinct angle sequence. Among those $2n$ angle sequences, at least one angle sequence must be minimum. If this minimum angle sequence belongs to $\mathcal{AS}(r)$, then r is selected as the nominee. So the initial configuration must have at least one nominee.

Now, let it be assumed that there are more than two nominees in the initial configuration. Without loss of generality let there be three nominees in the initial configuration. Let the first nominee whose angle sequence is minimum, has the minimum angle sequence in a particular direction say, $\mathcal{D}$. Then the second nominee must get its minimum angle sequence in the direction $\mathcal{D}'$, the opposite direction of $\mathcal{D}$ (as no two robots can have minimum angle sequence in the same direction). Now, the third nominee must have its minimum angle sequence in the direction of either $\mathcal{D}$ or $\mathcal{D}'$. But this can't be possible because, in a particular direction, no two robots have the same angle sequence. So there can not be more than two nominees in the initial configuration. $\square$
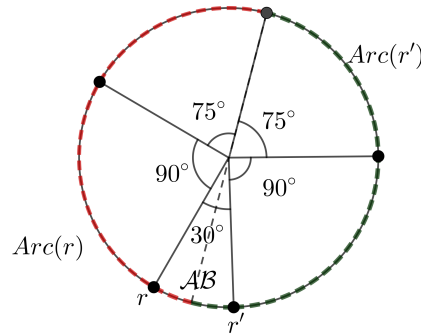

Fig. 2: A double nominee configuration where both r and r' are nominees. The angle bisector $\mathcal{AB}$ contains a robot. $Arc(r)$ is highlighted with red dotted line and $Arc(r')$ is highlighted with green dotted line.

Definition 4 (Single nominee configuration). *This is a configuration where there is only one nominee.*

Definition 5 (Double nominee configuration). *This is a configuration where there are two nominees in the configuration.*

Definition 6 (Angle Bisector in a double nominee configuration). *Let r and r' be two nominees in a double nominee configuration. The angle bisector*

of this configuration is defined as the straight line that bisects the angles formed by the robots r and r' and is denoted as $\mathcal{AB}$.

In the future, the term "angle bisector of a configuration" or the symbol $\mathcal{AB}$ will always be used for a double nominee configuration even if it is not mentioned explicitly.

Definition 7 (Arc of a nominee in a double nominee configuration).
Let r and r' be two nominees in a double nominee configuration. Let $\mathcal{AB}$ be the angle bisector of the angle between r and r'. Now, $\mathcal{AB}$ divides the circle into two arcs. Among these two arcs, the arc on which the robot r is located except the points of $\mathcal{AB}$ is called the arc of the robot r and is denoted as $Arc(r)$.

Definition 8 (Leader). *We call a robot, say r, the leader if any one of the following statements is true for the robot r:*

1. *r is the nominee in a single nominee configuration.*
2. *r is a nominee in a double nominee configuration and $Arc(r)$ has more robot than $Arc(r')$ where r' is another nominee.*

Definition 9 (Leader configuration). *A configuration is called a leader configuration if it has a leader.*

Definition 10 (Pivotal Direction). *In a leader configuration, the direction in which the leader has the minimum angle sequence is called the pivotal direction. The pivotal direction is denoted as $\mathcal{D}_p$.*
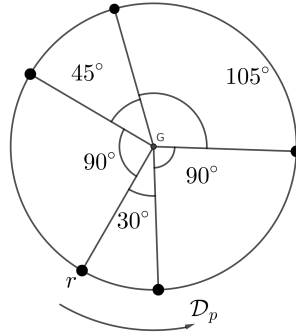

Fig. 3: Here the configuration is a leader configuration and r is the leader. The direction $\mathcal{D}_p$ is the direction in which r has the minimum angle sequence.

Proposition 3. *If an asymmetric configuration with an odd number of robots is not a leader configuration then the configuration must be a double nominee configuration with exactly one robot on the angle bisector $\mathcal{AB}$.*

Proof. Let C be an asymmetric configuration then by Proposition 2 C must be either a single nominee configuration or a double nominee configuration. By Definition 8 and Definition 9 any single nominee configuration must be a leader configuration. So, C must be a double nominee configuration. let r and r' be the two nominees in C. Let $\mathcal{AB}$ be the angle bisector of the configuration C. Now since the configuration C is not a leader configuration, the number of robots on $Arc(r)$ = number of robots on $Arc(r')$. Now if there is no robot or two robots on $\mathcal{AB}$ then the total number of the robots becomes even in the configuration, contrary to our assumption. So, $\mathcal{AB}$ must contain exactly one robot.   □

Definition 11 (Move Ready Robot). *In a leader configuration, let r be the first robot from leader, say r_0 in the direction $\mathcal{D}_p$ which satisfies the following condition:*

1. *r is not the first or second neighbour of leader r_0 in the direction $\mathcal{D}_p$.*
2. *If D is the destination of r in direction $\mathcal{D}$ and r' be the neighbour of r in the direction $\mathcal{D}$. Then $(R, R')_{\mathcal{D}} - (R, D)_{\mathcal{D}} > \alpha_1$, where α_1 is the angular distance between first and second neighbour of r_0 in the direction $\mathcal{D}_p$ and R and R' are the locations of r and r' respectively on the circle.*

Then r is defined as the Move Ready robot of the configuration

5   Algorithm *APF_CIRCLE*

Let $r_0, r_1, \ldots r_{n-1}$ be n robots (n is odd and $n > 2$) on a continuous circle $\mathcal{CIR}$. The robots can move freely on $\mathcal{CIR}$. Let for a robot r_i the location of r_i on $\mathcal{CIR}$ is denoted as R_i. Let initially the configuration is rotationally asymmetric. Let $\mathcal{D}$ be the direction in which r_{i+1} is a neighbor of r_i. Also, let $(R_i, R_{i+1})_{\mathcal{D}} = \alpha_i$ where all indices are considered in modulo n. Note that, $\sum_{i=0}^{n-1} \alpha_i = 2\pi$. Now, each robot has been provided with a sequence of angle $\beta_1, \beta_2, \ldots, \beta_{n-1}$ as input such that, $\sum_{i=0}^{n-1} \beta_i = 2\pi$. The robots are oblivious, silent, and have no chirality. The circle is not oriented. In this setting, the algorithm provided in this section guides the robots to form the sequence of angles provided to them on the circle $\mathcal{CIR}$.

The first challenge to do so is to find a robot and a direction from which this formation will start. This will be done by leader election. Now since the robots are oblivious the next problem that arises is to make the leader robot remain leader and the direction in which the pattern will be formed (pivotal direction) fixed throughout the execution of the algorithm. All of these challenges are considered and handled in the provided algorithm. In the following, the description of the algorithm and its correctness are discussed in detail.

5.1   Description and Correctness of *APF_CIRCLE*

Let $\mathcal{C}(0)$ be the initial configuration. According to Proposition 2, there can be at most two nominees and at least one nominee in the configuration $\mathcal{C}(0)$. If there

is a single nominee then it is the leader and the configuration is a leader config-
uration (Definition 8 and Definition 9). Otherwise, there must be two nominees
in $\mathcal{C}(0)$. Let r_1 and r_2 be two nominees in $\mathcal{C}(0)$. Now there are two possibili-
ties. Either $Arc(r_1)$ and $Arc(r_2)$ have same number of robots or they don't. Let,
$Arc(r_1)$ and $Arc(r_2)$ do not have equal number of robots then without loss of
generality Let $Arc(r_1)$ has more robots. In this case, r_1 would be the leader and
the configuration is again a leader configuration. So, let us consider the case
where $Arc(r_1)$ and $Arc(r_2)$ have equal numbers of robots. According to Proposi-
tion 3, there must be exactly one robot on the angle bisector $\mathcal{AB}$. Let us denote
that robot as r_d. Thus, when $\mathcal{C}(0)$ has two nominees and it is not a leader con-
figuration then, the robot r_d moves a positive angular distance ϵ_d towards any
of its neighbor, say r, in the direction $\mathcal{D}$ such that, $0 < \theta - \epsilon_d < \alpha_0$ (α_0 is the
smallest angle in the initial configuration and $\theta = (R_d, R)_{\mathcal{D}}$) and also, after r_d
moves the angular distance ϵ_d towards r, there is no robot on the angle bisector
of r_d and r (Fig.4). In the following two lemmas, we first prove the existence of
such ϵ_d and then prove that this move of r_d converts the configuration into a
leader configuration.

Lemma 1. *If the initial configuration is not a leader configuration then, the
robot r_d on the angle bisector $\mathcal{AB}$ always finds an $\epsilon_d > 0$ such that after moving ϵ_d
towards a neighbor r in the direction $\mathcal{D}$, there is no robot on the angle bisector of
R_d and R. Also if in the initial configuration $(R_d, R)_{\mathcal{D}} = \theta$ then, $0 < \theta - \epsilon_d < \alpha_0$.*

Proof. Let initially the angle bisector $\mathcal{AB}$ intersect the circle $\mathcal{CIR}$ at two points
A and B. Without loss of generality let r_d be on A. Without loss of generality
let r_d decides to move towards its neighbor r in the direction say $\mathcal{D}$. Let initially
$(R_d, R)_{\mathcal{D}} = \theta \geq \alpha_0$ as α_0 is the minimum angle of $\mathcal{C}(0)$. We will show that there
exists a positive $\epsilon_d < \theta$ such that after r_d moves ϵ_d towards r, in the direction
$\mathcal{D}$, r_d and r will not have any robot on the angle bisector of the angle formed by
the points R_d and R. Also, angular distance between R_d and R will be strictly
less than α_0 after the move of r_d (i.e., $\theta - \epsilon_d < \alpha_0$). Let P_α and Q_α be the points
of intersection of the circle $\mathcal{CIR}$ and the angle bisector of the angle formed by
R_d and R, after r_d moves towards r by an angular distance α in the direction $\mathcal{D}$.
Here without loss of generality, it is assumed that P_α is the point on the major
arc formed by R_d and R and Q_α is the point on the corresponding minor arc
(Fig.4).

Case 1: First, let us consider the case when $\theta = \alpha_0$. Let, $0 < \alpha < \alpha_0$. Now,
if Q_α is empty then we are done. So, let us assume for all positive $\alpha < \alpha_0$, Q_α
is non-empty. Now consider the one-one correspondence $\alpha \mapsto Q_\alpha$. Now since α
can have infinitely many values, there must be infinitely many Q_α which must
be occupied by a robot in the configuration contrary to our assumption that
there are finitely many robots on the circle $\mathcal{CIR}$. Hence, there must be some α,
say ϵ_d, in the open interval $(0, \alpha_0) = (0, \theta)$ such that, Q_{ϵ_d} is empty. Now since
$\theta > \epsilon_d > 0 = \theta - \alpha_0$, the condition, $0 < \theta - \epsilon_d < \alpha_0$ is satisfied.

Case 2: Now let us consider the case when $\theta > \alpha_0$. Here ϵ_d must be in the
open interval $(\theta - \alpha_0, \theta)$ for the condition $0 < \theta - \epsilon_d < \alpha_0$ to satisfy. Now let
us consider the one-one correspondence $\alpha \mapsto Q_\alpha$ that maps all points which are

at an angular distance $\alpha \in (\theta - \alpha_0, \theta)$. Again by a similar argument as in the previous case it can be shown that there is an α, say $\epsilon_d \in (\theta - \alpha_0, \theta)$ such that Q_{ϵ_d} is empty. Also since $\theta - \alpha_0 < \epsilon_d < \theta$, the condition $0 < \theta - \epsilon_d < \alpha_0$ is also satisfied. Hence the result.   □
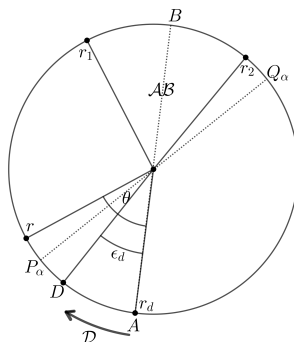

Fig. 4: r_d is the robot on $\mathcal{AB}$ of the double nominee configuration with two nominees r_1 and r_2. Here in this scenario the robot r_d moves an angular distance ϵ_d to D.

Corollary 1 *If the initial configuration is not the leader configuration, then within one epoch the configuration will be the leader configuration.*

Proof. Since $\mathcal{C}(0)$ is not a leader configuration, the robot r_d on $\mathcal{AB}$ can move an angular distance $\epsilon_d > 0$ satisfying the condition $0 < \theta - \epsilon_d < \alpha_0$, towards its neighbor r in the direction $\mathcal{D}$ such that after the move, the angle bisector of the angle formed by R_d and R does not contain any robot (By Lemma 1). Here, $\theta = (R_d, R)_{\mathcal{D}}$. Now note that after this move by r_d, if the configuration is a single nominee configuration then, by Definition 8 and Definition 9, the configuration must be a leader configuration. Otherwise, if the configuration is not a single nominee configuration then, r_d and r must be the two nominees in the new configuration. But since they do not have any robot on $\mathcal{AB}$, the configuration must be a leader configuration.   □

Thus, if the initial configuration is not a leader configuration, it will become one within one epoch.

Taget embedding: To embed the target pattern on the circle $\mathcal{CIR}$, a point and a direction on $\mathcal{CIR}$ must be agreed upon by all the robots. let T_0 be the point and $\mathcal{D}$ be the direction. Let the j-th target locations on $\mathcal{CIR}$ from T_0 in the direction $\mathcal{D}$ is denoted by T_j, where $j \in \{0, 1, \ldots, n-1\}$. The points are embedded in such a way on $\mathcal{CIR}$ that $(T_j, T_{j+1})_{\mathcal{D}} = \beta_j$ where the sequence $\beta_0, \beta_1, \ldots, \beta_{n-1}$ is lexicographically smallest upto rotation of the input pattern given to the robots (all the indices are considered in modulo n).
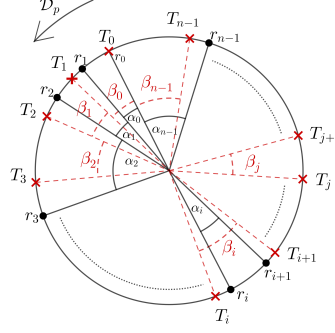
Fig. 5: $(\beta_0, \beta_1, \ldots \beta_{n-1})$ is the smallest in lexicographic ordering of all possible sequences that can be formed from the input upto rotation. The sequence is embedded on the circle starting from the location of r_0 and in the direction $\mathcal{D}_p$.

In a leader configuration, the location of leader r_0, denoted as R_0 serves the purpose of T_0. We also claim that the pivotal direction $\mathcal{D}_p$ of r_0 must be unique, which serves the purpose of the direction on which all robots can agree. Thus, the target locations T_j can be embedded on $\mathcal{CIR}$ as described above (Fig. 5).

The next lemma ensures the claim that the leader r_0 must have a unique pivotal direction $\mathcal{D}_p$.

Lemma 2. *In a leader configuration, the pivotal direction is unique.*

Proof. If possible let, the leader r_0 has two pivotal directions $\mathcal{D}$ and $\mathcal{D}'$. Then both of $\mathcal{AS}_\mathcal{D}(r_0)$ and $\mathcal{AS}_{\mathcal{D}'}(r_0)$ are minimum and equal. Let, R_1 and R_{n-1} are the location of the neighbours of r_0 in the direction $\mathcal{D}$ and $\mathcal{D}'$ respectively. Then both the angles $(R_0, R_1)_\mathcal{D}$ and $(R_0, R_{n-1})_{\mathcal{D}'}$ must be α_0, where α_0 is the minimum angle of the configuration. In this case $\mathcal{AS}_\mathcal{D}(r_{n-1})$ and $\mathcal{AS}_{\mathcal{D}'}(r_1)$ both are strictly less than $\mathcal{AS}_\mathcal{D}(r_0) = \mathcal{AS}_{\mathcal{D}'}(r_0)$. So r_0 can't be the leader in the first place, which is not possible. Thus the pivotal direction of the leader must be unique. □

Now in a leader configuration, let us denote the i-th robot from r_0, (i.e., the leader) in the direction $\mathcal{D}_p$ as r_i and $(R_i, R_{i+1})_{\mathcal{D}_p}$ as α_i, where all the indices are considered in modulo n. Note that $\alpha_0 \leq \alpha_i$, for all i. Now, if there exists some $i \neq 0$ for which $\alpha_i = \alpha_0$ or, $\alpha_0 \geq \beta_0$ (β_0 is the smallest angle in the input target pattern according to the embedding), i.e., if $\alpha_0 \geq \min_{i \neq 0}\{\alpha_i, \beta_0\}$ then, the leader r_0 can always move in the direction $\mathcal{D}_p$, a positive angular distance $\epsilon < \alpha_0$ in such a way that $\alpha_0 - \epsilon < \min_{i \neq 0}\{\alpha_i, \beta_0\}$ and there is no robot on the angle bisector of the angle formed by R_0 and R_1, after r_0 moves (Fig.6). Observe that, $(R_0, R_1)_{\mathcal{D}_p} = \alpha_0 - \epsilon$, after r_0 moves which is strictly less than all other angles in the current configuration (Here, $\alpha_0 = (R_0, R_1)_{\mathcal{D}_p}$ before r_0 moves). In the following lemma, we have proved the existence of such ϵ. Note that after r_0

moves the configuration remains a leader configuration though the leader and the pivotal direction might change. Note that, if the leader changes then, it must be none other than the robot r_1 in the configuration before r_0 moved. But after the move we can assure that the new α_0 must be strictly less than $\min_{i \neq 0}\{\alpha_i, \beta_0\}$ i.e., strictly less than any other angle except the new α_0 in the current configuration as well as the minimum angle of the target configuration.
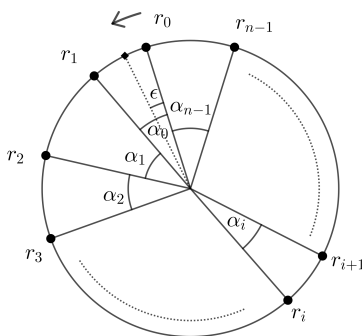

Fig. 6: Here $\alpha_i = \alpha_0$, so the leader r_0 moves towards r_1 and angular distance ϵ such that after the move the angular distance between r_0 and r_1 is strictly smallest in the current configuration and the inputs, also after moving the angle bisector of the angle formed by r_0 and r_1 doesn't contain any robot.

Lemma 3. *In a leader configuration, if $\alpha_0 \geq \min_{i \neq 0}\{\alpha_i, \beta_0\}$ where α_0 is the angle between the leader r_0 and the neighbour of r_0, say r_1, in the direction $\mathcal{D}_p$ then, there exists a positive $\epsilon < \alpha_0$ such that $\epsilon > \alpha_0 - \min_{i \neq 0}\{\alpha_i, \beta_0\}$ and after moving ϵ angular distance towards r_1, the angle bisector of r_0 and r_1 does not contain any other robot.*

Proof. Let us consider the open interval $(\alpha_0 - \min_{i \neq 0}\{\alpha_i, \beta_0\}, \alpha_0)$. Now for any $\alpha \in (\alpha_0 - \min_{i \neq 0}\{\alpha_i, \beta_0\}, \alpha_0)$, let $P_\alpha Q_\alpha$ be the angle bisector of the angle formed by R_0 and R_1, after r_0 moves an angular distance of α towards r_1. without loss of generality, it is assumed that P_α is the point on the major arc joining R_0 and R_1 and Q_α be the same on the minor arc after the move of r_0 by an angular distance of α towards r_1. Now considering the one-one correspondence $\alpha \mapsto Q_\alpha$ and with similar arguments as in Lemma 1 we can prove that there is a $\alpha \in (\alpha_0 - \min_{i \neq 0}\{\alpha_i, \beta_0\}, \alpha_0)$, say ϵ, such that the angle bisector of the angle formed by R_0 and R_1 after the move of r_0, does not have any robot on it. Also, since $\in (\alpha_0 - \min_{i \neq 0}\{\alpha_i, \beta_0\}, \alpha_0)$, the conditions $\epsilon < \alpha_0$ and $\epsilon > \alpha_0 - \min_{i \neq 0}\{\alpha_i, \beta_0\}$ both are satisfied. □

So, from a leader configuration within at most one epoch, the configuration changes into another leader configuration where the smallest angle α_0 of the configuration is unique and also strictly less than β_0, the smallest angle of the target configuration. In this configuration if $\alpha_1 > \alpha_0$ is strictly less than all other α_j for all $j \neq 0, 1$ and also strictly less than β_0, i.e., $\alpha_1 < \min_{j \neq 0,1}\{\alpha_j, \beta_0\}$ then we can ensure that the leader r_0 and the pivotal direction $\mathcal{D}_p$ never changes even if the robots $r_3, r_4, \ldots, r_{n-1}$ moves according to the algorithm 1. This is because Algorithm APF_CIRCLE ensures that no angle less or equal to α_1 is formed during the moves by the above-mentioned robots.

Now if in a leader configuration $\alpha_0 < \min_{i \neq 0}\{\alpha_i, \beta_0\}$ but $\alpha_1 \geq \min_{j \neq 0,1}\{\alpha_j, \beta_0\}$ then, r_2 can always move an angular distance ϵ_1 towards r_1 such that the resultant configuration is again a leader configuration and $\mathcal{D}_p$ does not change due to this movement and also, the new α_1 in the resultant configuration after the move by r_2 must be strictly less than all α_j and β_0 where $j \neq 0, 1$. So, the angle $(R_1, R_2)_{\mathcal{D}_p}$, after the move by r_2 becomes old $\alpha_1 - \epsilon_1$ which is strictly less than $\min_{i \neq 0,1}\{\alpha_i, \beta_0\}$ (Fig.7). The following lemma guarantees the existence of such ϵ_1.
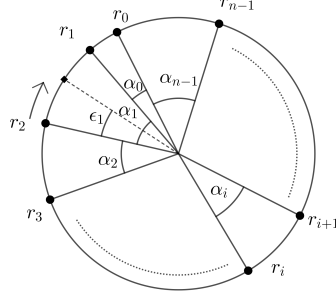

Fig. 7: Here α_1 is not strictly smaller than other α_is (except $i = 0$) or β_js. So r_2 moves an angular distance ϵ_1 towards r_1 such that the new α_1 after the move becomes the second uniquely minimum angle of the configuration and also less than all β_js.

Lemma 4. *If in a leader configuration $\alpha_0 < \min_{i \neq 0}\{\alpha_i, \beta_0\}$ and $\alpha_1 \geq \min_{j \neq 0,1}\{\alpha_j, \beta_0\}$ then there exists $\epsilon_1 > 0$ such that $\alpha_1 - \min_{j \neq 0,1}\{\alpha_j, \beta_0\} < \epsilon_1 < \alpha_1 - \alpha_0$*

Proof. From the given conditions,

$$\alpha_0 < \min_{i \neq 0}\{\alpha_i, \beta_0\} \ \& \ \alpha_1 \geq \min_{j \neq 0,1}\{\alpha_j, \beta_0\}. \tag{1}$$

We have to find an $\epsilon_1 > 0$ such that it follows the following conditions:

$$\alpha_0 < \alpha_1 - \epsilon_1 \ \& \ \alpha_1 - \epsilon_1 < \min_{j \neq 0,1}\{\alpha_j, \beta_0\}.$$

Let us consider, $\epsilon_1 = \alpha_1 - \frac{\alpha_0 + \min\limits_{i \neq 0}\{\alpha_i, \beta_j\}}{2}$. Then from the inequality in (1), $\frac{\alpha_0 + \min\limits_{i \neq 0}\{\alpha_i, \beta_j\}}{2} > \alpha_0$ and thus $\alpha_1 - \epsilon_1 > \alpha_0$. Hence first part is proved.

To prove the second part, if possible let,

$$\alpha_1 - \epsilon_1 \geq \min_{i \neq 0,1}\{\alpha_i, \beta_0\}$$

Also since $\min\limits_{i \neq 0}\{\alpha_i, \beta_0\} \leq \min\limits_{j \neq 0,1}\{\alpha_j, \beta_0\}$. We have the following inequality,

$$\frac{\alpha_0 + \min\limits_{i \neq 0}\{\alpha_i, \beta_j\}}{2} = \alpha_1 - \epsilon_1 \geq \min_{j \neq 0,1}\{\alpha_j, \beta_0\} \geq \min_{i \neq 0}\{\alpha_i, \beta_0\}$$

Which implies, $\alpha_0 \geq \min\limits_{i \neq 0}\{\alpha_i, \beta_0\}$. Which is a contradiction to the inequality in (1). Thus $\alpha_1 - \epsilon_1 < \min\limits_{j \neq 0,1}\{\alpha_j, \beta_0\}$ □

So, even after r_2 moves an angular distance ϵ_1 towards r_1, the configuration still remains a leader configuration which satisfies the conditions that, $\alpha_0 < \min\limits_{i \neq 0}\{\alpha_i, \beta_0\}$ and $\alpha_1 < \min\limits_{j \neq 0,1}\{\alpha_j, \beta_0\}$. We call this kind of configuration a RFC. The definition of RFC has been mentioned formally in the following:

Definition 12 (Ready to Form Configuration (RFC)). *A leader configuration is said to be a Ready to Form configuration, or a RFC, if it satisfies the conditions $\alpha_0 < \min\limits_{i \neq 0}\{\alpha_i, \beta_0\}$ and $\alpha_1 < \min\limits_{j \neq 0,1}\{\alpha_j, \beta_0\}$.*

Other than RFC we have another type of configuration called PFC that need to be defined first.

Next when RFC is achieved the Move Ready robot r_i ($i \in 3, 4, \ldots, n-1$) moves to its corresponding target destination T_i and terminates. Note that since $\alpha_0 < \alpha_1 < \min\limits_{j \neq 0,1}\{\alpha_j, \beta_0\}$ and the move of r_i does not make any angle less or equal to α_1, the configuration still remains a RFC and the leader and pivotal direction $\mathcal{D}_p$ does not change. We now show that from an RFC within at most $n-3$ epochs, the configuration must become a PFC. For that, we need to prove the following lemmas.

Lemma 5. *In a RFC if a robot r_i, $i \geq 3$ is not a Move Ready robot and the destination of r_i i.e., T_i is in the direction $\mathcal{D}$ from R_i then, the neighbor of r_i in the direction $\mathcal{D}$, say r_k, must also have its target destination T_k in direction $\mathcal{D}$ from R_k.*

Proof. Let r_i ($i \geq 3$) be a robot that is not Move Ready and its destination T_i is in direction $\mathcal{D}$ from R_i. Let r_k be the neighbour of r_i in the direction $\mathcal{D}$ (k can be either $i + 1$ or, $i - 1$ in modulo n). Since r_i is not move

(a) Ready to Form Configuration. Here $\alpha_0 < \min\limits_{i \neq 0}\{\alpha_i, \beta_0\}$ and $\alpha_1 < \min\limits_{j \neq 0,1}\{\alpha_j, \beta_0\}$.

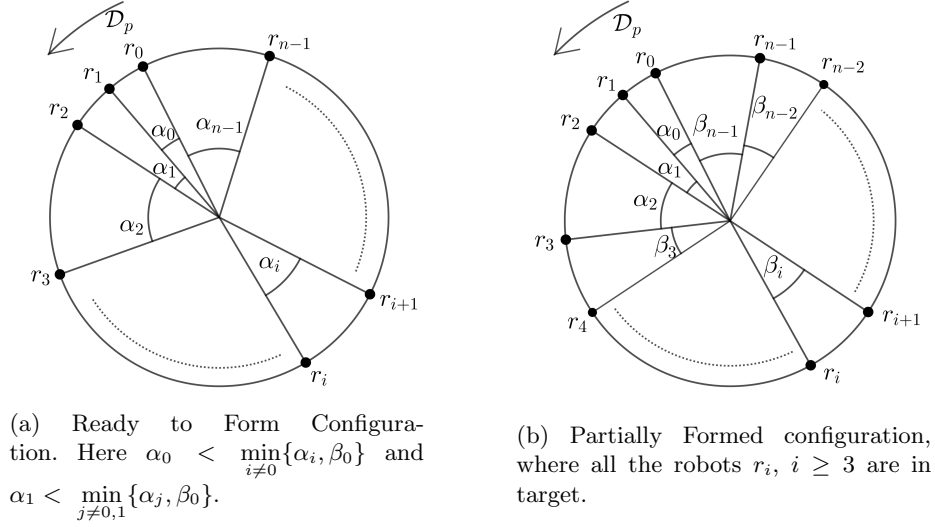(b) Partially Formed configuration, where all the robots r_i, $i \geq 3$ are in target.

Fig. 8: RFC and PFC

ready, $(R_i, R_k)_{\mathcal{D}} - (R_i, T_i)_{\mathcal{D}} \leq \alpha_1$. Now there can be two possibilities. Either, $(R_i, R_k)_{\mathcal{D}} - (R_i, T_i)_{\mathcal{D}} \leq 0$ or, $0 < (R_i, R_k)_{\mathcal{D}} - (R_i, T_i)_{\mathcal{D}} \leq \alpha_1$. Now, if possible let the destination of r_k i.e., T_k be in the direction $\mathcal{D}'$ from R_k.

Case 1: Let $(R_i, R_k)_{\mathcal{D}} - (R_i, T_i)_{\mathcal{D}} \leq 0$. This implies T_i is further than R_k in the direction $\mathcal{D}$, from R_i (Fig.9a). Now consider $k = i+1$ and thus the $\mathcal{D} = \mathcal{D}_p$. Note that i then can not be $n-1$ as T_{n-1} can not be further than $R_0 = T_0$ in the direction $\mathcal{D}_p$ from R_{n-1} according to the target embedding. Now, for all other values for $i \geq 3$, if T_{i+1} is in the direction $\mathcal{D}'_p$ from R_{i+1}, then T_{i+1} appears before T_i in the direction $\mathcal{D}_p$ in the embedding which is contradiction. Similarly, let us consider $k = i-1$ and thus $\mathcal{D} = \mathcal{D}'_p$. Here note that i can not be 3 as otherwise r_3 is Move Ready. This is because T_3 and T_2 must be on the arc joining from R_2 to R_3 in the direction $\mathcal{D}_p$. Thus T_3 can not be further than R_2 from R_3 in the direction $\mathcal{D}'_p$ as, $(T_3, R_2)_{\mathcal{D}'_p} > (T_3, T_2)_{\mathcal{D}'_p} = \beta_2 > \alpha_1 > 0$. Now for all other values of $i > 3$, it can be shown that we will arrive at a contradiction by a similar argument as in the case where $k = i+1$ has been considered.

Case 2: Let $0 < (R_i, R_k)_{\mathcal{D}} - (R_i, T_i)_{\mathcal{D}} \leq \alpha_1$. This implies R_k is further than T_i from R_i in the direction $\mathcal{D}$ but, $(T_i, R_k)_{\mathcal{D}} \leq \alpha_1$ (Fig. 9b). Let $k = i+1$ and hence $\mathcal{D} = \mathcal{D}_p$ (i can not be $n-1$ as shown earlier in case 1). Now, according to the embedding T_i can not be further than T_{i+1} from T_0 in the direction $\mathcal{D}_p$. Hence, T_{i+1} must be on the arc joining from T_i to R_{i+1} in the direction $\mathcal{D}_p$. This implies $\beta_0 \leq (T_i, T_{i+1})_{\mathcal{D}_p} \leq (T_i, R_{i+1})_{\mathcal{D}_p} \leq \alpha_1 \implies \beta_0 \leq \alpha_1$, a contradiction due to the fact that the configuration is a RFC. Similarly if $k = i-1$ and hence the direction $\mathcal{D} = \mathcal{D}'_p$ then again we will arrive at a contradiction by a similar argument.
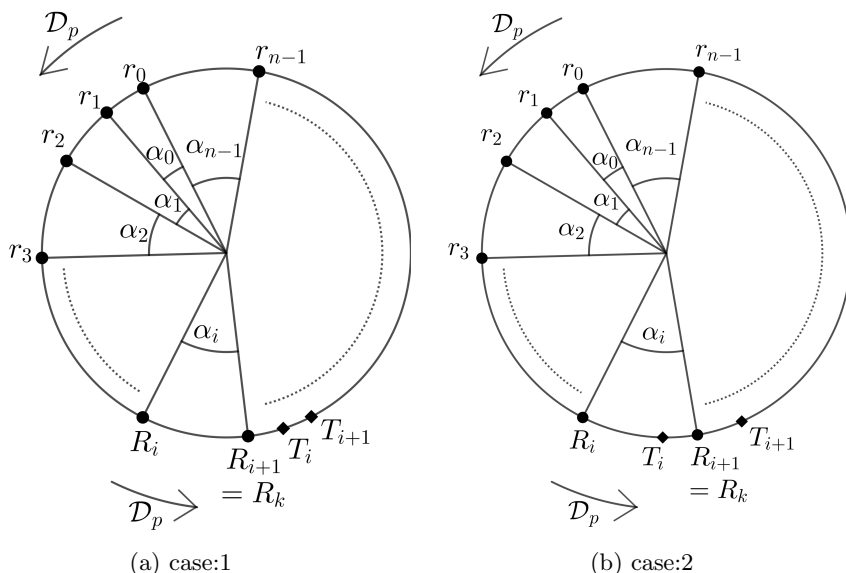
(a) case:1   (b) case:2

Fig. 9: If the target destination T_i of the robot r_i is in the direction $\mathcal{D}$, then the target destination T_{i+1} of its neighbour r_{i+1} is also in the same direction $\mathcal{D}$.

Since for both the possibilities we arrive at a contradiction, T_k must also be in the direction of $\mathcal{D}$ from R_k.   □

Lemma 6. *If a RFC, is not a PFC then there exists a robot r_p which is Move Ready.*

Proof. A robot is called terminated if it has already reached its target. If possible let in a RFC the robots r_i $(i \geq 3)$ are either terminated or not Move Ready. Let r_k be a robot from R_0 in the direction $\mathcal{D}_p$ which has not terminated and is not Move Ready. Let the target of r_k i.e., T_k be in a direction $\mathcal{D}$ from R_k. Observe that if $r_k = r_3$, then $\mathcal{D} = \mathcal{D}_p$. Otherwise, since T_2 is in the direction $\mathcal{D}_p$ from R_2, $(R_3, R_2)_{\mathcal{D}'_p} - (R_3, T_3)_{\mathcal{D}'_p} = (R_2, T_3)_{\mathcal{D}_p} \geq (T_2, T_3)_{\mathcal{D}_p} = \beta_2 > \alpha_1$ and hence r_3 becomes Move Ready. Similarly if $r_k = r_{n-1}$ then, T_{n-1} must be in the direction $\mathcal{D}'_p$ from R_{n-1}. Otherwise, T_{n-1} must lie on the arc joining R_{n-1} and $T_0 = R_0$ in the direction $\mathcal{D}_p$ which implies $(R_{n-1}, R_0)_{\mathcal{D}_p} - (R_{n-1}, T_{n-1})_{\mathcal{D}_p} = (T_{n-1}, T_0)_{\mathcal{D}_p} = \beta_{n-1} > \alpha_1$ a contradiction.

Now we claim that, For a robot $r_i, (i \geq 3)$ which has not terminated and is not move ready, if the direction of its target is in the direction $\mathcal{D}$ from r_i, then the neighbor of r_i , say r_j in the direction $\mathcal{D}$ must have not terminated also. Otherwise, if r_j is terminated then it must be on T_j. Also, T_i must be on the arc joining the points from R_i to T_j in the direction $\mathcal{D}$. This implies $(R_i, R_j)_{\mathcal{D}} - (R_i, T_i)_{\mathcal{D}} = (T_i, T_j)_{\mathcal{D}} = \beta_t > \alpha_1$ $(t \in \{i, j\})$ and thus r_i becomes move ready contrary to the assumption.

So, now for a robot r_{k_1} which is not Move Ready and has not terminated yet, let $\mathcal{D}$ be the direction of T_{k_1} from R_{k_1}. Also let r_{k_2} be the neighbour of r_{k_1} in the direction $\mathcal{D}$. By Lemma 5 and the above claim r_{k_2} must have not terminated yet and the direction of T_{k_2} must be in the direction $\mathcal{D}$ from R_{k_2}. Now by mathematical induction, it can be shown that all robots r_i ($i \in \{3, 4, \ldots, n-1\}$) in the direction $\mathcal{D}$ from r_k, must have not terminated and are not Move Ready. So either r_3 or r_{n-1} must be not Move Ready and has not terminated. if r_3 is not move ready and has not terminated then the direction of T_3 must be $\mathcal{D}_p$ from R_3 and then by induction it can be shown that r_{n-1} must also be not Move ready and has not terminated and direction of T_{n-1} must be in $\mathcal{D}_p$ from R_{n-1} which is a contradiction. Similarly, if r_{n-1} is not Move ready and has not terminated then T_{n-1} must be in the direction $\mathcal{D}'_p$ from R_{n-1} which will imply r_3 is not Move Ready and is not terminated and T_3 must be in direction $\mathcal{D}'_p$ from R_3. which is again a contradiction. Hence the result.

□

Now as a direct corollary of the above Lemma 6, we can conclude the following.

Corollary 2 *If a RFC is not a PFC then, in each epoch at least one of the robots, say r_p, $p \geq 3$ must reach its target T_p.*

Now we are in a shape to establish the claim of a RFC becoming a PFC within $n-3$ epochs using the above Corollary 2. Since in each epoch, at least one of r_i ($i \geq 3$) is moving to its target T_i, and there can be at most $n-3$ such robots, we can ensure that within at most $n-3$ epochs, an RFC must become a PFC. We formalize this in the following theorem.

Theorem 1. *From the initial configuration, within n epochs the configuration will become a PFC.*

Proof. From initial configuration, RFC is achieved in at most 3 epochs and from RFC to PFC within $n-3$ epochs. Hence the theorem.   □

Now when the PFC has been achieved, then only r_1 and r_2 are not in T_1 and T_2. Note that since $\alpha_1 < \min_{j \neq 0,1}\{\alpha_j, \beta_0\} \leq \beta_1$, r_1 can not move even if it is activated earlier or together with r_2. On the other hand when activated r_2 first checks if its move to T_2 might change the leader and the pivotal direction. Note that if r_2 finds that $\beta_0 + \beta_1 - \alpha_0 < \beta_{n-1}$ then moving to its target T_2 does not change the leader and the pivotal direction. So, in this case, it moves to T_2. Otherwise, r_2 moves to a point, say D_2 such that $(R_1, D_2)_{\mathcal{D}_p} = \beta_{n-1} - \delta$, where $\delta \in (0, \beta_{n-1} - \beta_0 + \alpha_0)$ (Fig.10). In the following lemma, we have justified that any of such moves by r_2 does not change the leader and the pivotal direction.

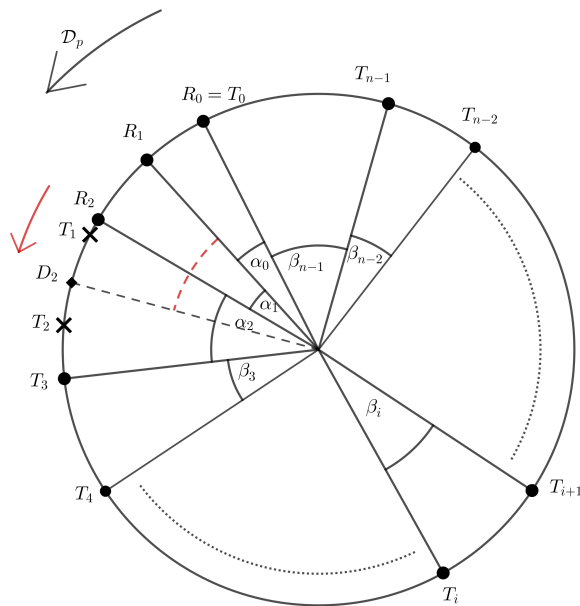**Lemma 7.** *In a PFC, after r_2 moves for the first time, the leader and its pivotal direction do not change.*

Fig. 10: If $\beta_0 + \beta_1 - \alpha_0 \geq \beta_{n-1}$, then r_2 moves to the point D_2 in $\mathcal{D}_p$ such that $(R_1, D_2)_{\mathcal{D}_p}$ marked here with red arc, equals to $\beta_{n-1} - \delta$.

Proof. In a PFC, r_2 always moves before r_1 even if r_1 is activated earlier or together with r_2. When r_2 moves for the first time in a PFC, r_1 is not on T_1, and r_2 moves to D_2. Now, either $D_2 = T_2$ or, it is a point on the circle such that, $(R_1, D_2)_{\mathcal{D}_p} = \beta_{n-1} - \delta$, where $0 < \delta < \beta_{n-1} - \beta_0 + \alpha_0$. As $\alpha_0 < \alpha_1 < \beta_0 \leq \beta_1$, then the destination D_2 of r_2 must be on the arc joining from point R_2 to point $R_3 = T_3$ in the direction $\mathcal{D}_p$.

Case-I: If $(R_1, T_2)_{\mathcal{D}_p} = \beta_0 + \beta_1 - \alpha_0 < \beta_{n-1}$, then r_2 moves to its target T_2. Then the angle sequence of r_0 in the pivotal direction remains uniquely minimum, as $(R_1, T_2)_{\mathcal{D}_p} < \beta_{n-1}$. Thus in this case the leader and the pivotal direction will not change.

Case-II: If $(R_1, T_2)_{\mathcal{D}_p} = \beta_0 + \beta_1 - \alpha_0 \geq \beta_{n-1}$, then r_2 moves to a point D_2 in the direction $\mathcal{D}_p$ such that $(R_1, D_2)_{\mathcal{D}_p}$ must be $\beta_{n-1} - \delta$. Then the minimum angle sequence of the configuration remains unique and belongs to $\mathcal{AS}(r_0)$ and the pivotal direction also remains same as $\beta_{n-1} > \beta_{n-1} - \delta$, for any $\delta \in (0, \beta_{n-1} - \beta_0 + \alpha_0)$. □
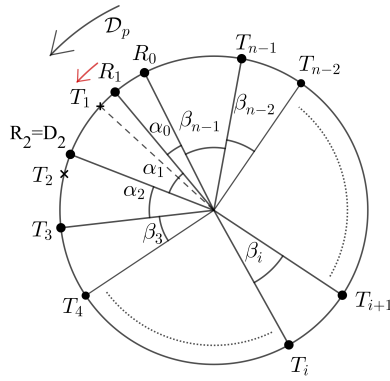
After r_2 moves for the first time in a PFC, r_1 will now move to its target T_1. Note that until r_1 reaches T_1, r_2 will not move even if it is activated according to the algorithm 1. It has to be ensured that after r_2 moves for the first time r_1 will move when activated and after r_1 moves to T_1 the leader and pivotal direction will remain the same. The following lemmas will establish these claims.

Lemma 8. *In a PFC, after r_2 moves for the first time, the condition $(R_1, R_2)_{\mathcal{D}_p} > \beta_1$ must becomes true.*

Proof. In a PFC, r_2 if moves for the first time either moves to T_2 or, moves to D_2 such that $(R_1, D_2)_{\mathcal{D}_p} = \beta_{n-1} - \delta$ for some $\delta \in (0, \beta_{n-1} - \beta_0 + \alpha_0)$.

    Case-I: Let r_2 has moved to T_2. We now have to show that $(R_1, T_2)_{\mathcal{D}_p} > \beta_1$. If $(R_1, T_2)_{\mathcal{D}_p} \leq \beta_1$ then, $\beta_0 + \beta_1 - \alpha_0 \leq \beta_1 \implies \alpha_0 \geq \beta_0$ which is a contradiction. Hence $(R_1, T_2)_{\mathcal{D}_p} > \beta_1$.

    Case-II: Now let us consider the case where r_2 has moved to D_2 such that $(R_1, D_2)_{\mathcal{D}_p} = \beta_{n-1} - \delta$, where $\delta \in (0, \beta_{n-1} - \beta_0 + \alpha_0)$. Observe that according to the target embedding $\beta_{n-1} \geq \beta_1$. This implies for any $\delta \in (0, \beta_{n-1} - \beta_0 + \alpha_0)$, $(R_1, D_2)_{\mathcal{D}_p} = \beta_{n-1} - \delta \geq \beta_1 - \delta > \beta_1$ and hence the result. $\qquad\square$

(a) After r_2 moves for the first time r_1 moves to its target without collision.
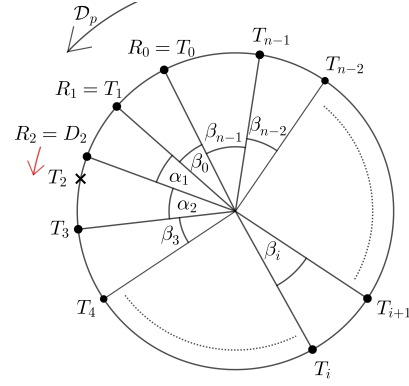
(b) When r_1 is on target r_2, if already not on target, moves to target T_2.

Fig. 11: r_1 and r_2 moves to target.

    In a PFC after r_2 moves for the first time, it will never move again until r_1 reaches its target T_1. By the Lemma 8, if r_2 moves for the first time, then the condition $(R_1, R_2)_{\mathcal{D}_p} > \beta_1$ must becomes true, which makes sure that r_1 can move to T_1 when activated again. We now ensure that to reach T_1, r_1 does not have to cross D_2. For that we have to show that $(T_1, D_2)_{\mathcal{D}_p} > 0$. If possible let, $(T_1, D_2)_{\mathcal{D}_p} = \beta_{n-1} - \delta - \beta_0 + \alpha_0 \leq 0$, then, $\delta \geq \beta_{n-1} - \beta_0 + \alpha_0$ contradicting the choice of δ. Hence to reach T_1, r_1 does not have to cross D_2. Thus r_1 moves to T_1 (Fig.11a). We now show that this move by r_1 does not change the leader and the pivotal direction if the target is already not formed.

Lemma 9. *If in a PFC, r_1 moves to its target T_1, then either pattern formation is complete or, the leader and the pivotal direction don't change.*

Proof. If r_2 already moved to T_2 during the first time it moved in a PFC then, after r_1 moves to T_1, the target is formed already. So let us consider, in a PFC,

r_2 during its first move moved to D_2 in such a way that $(R_1, D_2)_{\mathcal{D}_p} = \beta_{n-1} - \delta$, where $\delta \in (0, \beta_{n-1} - \beta_0 + \alpha_0)$. This implies $\beta_0 + \beta_1 - \alpha_0 \geq \beta_{n-1}$. Now, by Lemma 8, r_1 moves to T_1 and r_2 is on D_2. So after the move of r_1 the angle between them will be $(T_1, D_2)_{\mathcal{D}_p}$. If possible let the leader or the pivotal direction has changed by this move. This implies $(T_1, D_2)_{\mathcal{D}_p} = \beta_{n-1} - \delta - \beta_0 + \alpha_0 \geq \beta_1 \implies \beta_{n-1} - \delta \geq \beta_0 + \beta_1 - \alpha_0 \geq \beta_{n-1}$ which is a contradiction. Hence the leader and pivotal direction do not change.   □

Now if the pattern formation is already done then upon activation again, if a robot sees that the smallest angle sequence of the configuration is the given lexicographically smallest angle sequence (upto rotation) then it terminates. Note that after pattern formation is done within one epoch all robots will terminate. Now if the pattern is not formed even after r_1 moved in a PFC, that implies r_2 is not on T_2. Note that since the leader and the pivotal direction remain the same the target embedding also remains the same. In this scenario, upon activation, r_2 moves to T_2 (Fig.11b)target pattern is formed. So from a PFC within at most 4 more epochs, all the robots terminate knowing the pattern has been formed. Now, From all of these above descriptions, we can now conclude the following theorem.

Theorem 2. $n = 2k + 1$ *oblivious and silent robots placed on a circle can solve arbitrary pattern formation problem on a circle by executing the algorithm APF_CIRCLE within at most $n + 4$ epochs, even without chirality agreement under a semi-synchronous scheduler iff the initial configuration is rotationally asymmetric.*

6   Randomized APF for any $n \geq 3$

In the arbitrary pattern formation problem on a circle, if the leader configuration can be achieved for any n number of robots, then the algorithm APF_CIRCLE can solve the problem of pattern formation.

Now let there are $n = 2k$ numbers of robots on the circle in the initial configuration $\mathcal{C}(0)$. If there is a single nominee then the configuration is leader configuration. Otherwise, the configuration must be a double nominee configuration. Let, r_{n1} and r_{n2} be two nominees of the configuration $\mathcal{C}(0)$ which have their minimum angle sequences in the directions $\mathcal{D}_{n1}$ and $\mathcal{D}_{n2}$ respectively. $\mathcal{D}_{n1}$ and $\mathcal{D}_{n2}$ can be same or different. Let, $\mathcal{AB}$ be the angle bisector of the angle between r_{n1} and r_n2, which divides the circle into two arcs. Now if $Arc(r_{n1})$ contains more robots than $Arc(r_{n2})$, then $Arc(r_{n1})$ becomes the leader and leader configuration will be achieved. But if $Arc(r_{n1})$ and $Arc(r_{n2})$ contain the same number of robots then leader configuration can't be achieved from double nominee configuration.

Let us consider the case where $Arc(r_{n1})$ and $Arc(r_{n2})$ contain same number of robots. As the number of robots is even, then on the angle bisector $\mathcal{AB}$ either there are two robots or no robot. Then the two nominees r_{n1} and r_{n2} randomly

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Algorithm 1: *APF_CIRCLE*

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1   $r \leftarrow$ myself.;

2   **if** *the configuration is not leader configuration* **then**

3      $r_d \leftarrow$ robot on the angle bisector $\mathcal{AB}$;

4   **if** *I am r_d* **then**

5   move towards any neighbour by a small angle ϵ such that, $\epsilon < \alpha_0$ and r_d and its neighbours don,t have any other robot on their angle bisector;

6   **else**

7      $r_i \leftarrow$ the i-th robot from leader in the direction $\mathcal{D}_p$;

8      $T_i \leftarrow$ target destination of r_i;

9   **if** *I am the leader* **then**

10   **if** $\alpha_0 \geq \min\limits_{i \neq 0}\{\alpha_i, \beta_0\}$ **then**

11   choose $\epsilon \in (\alpha_0 - \min\limits_{i \neq 0}\{\alpha_i, \beta_0\})$ such that even after leader move ϵ angular distance towards r_1 the configuration remains a leader configuration ;

12   move by an angular distance ϵ towards r_1;

13   **else**

14   **if** *I am r_2* **then**

15   **if** *all robots r_p are in T_p, for all $p \geq 3$* **then**

16   **if** $\beta_0 + \beta_1 - \alpha_0 < \beta_{n-1}$ **then**

17   move to T_2;

18   **else**

19   **if** *r_1 is not on T_1* **then**

20   choose $\delta \in (0, \beta_{n-1} - \beta_0 + \alpha_0)$;

21   move to an angular distance $\alpha_0 + \beta_{n-1} - \delta$ from r_0 in direction $\mathcal{D}_p$;

22   **else**

23   move to T_2.;

24   **else**

25   **if** $\alpha_0 < \min\limits_{i \neq 0}\{\alpha_i, \beta_0\}$ **then**

26   **if** $\alpha_1 \geq \min\limits_{j \neq 0,1}\{\alpha_j, \beta_0\}$ **then**

27   choose $\epsilon_1 \in (\alpha_1 - \min\limits_{j \neq 0,1}\{\alpha_j, \beta_0\}, \alpha_1 - \alpha_0)$;

28   move by an angular distance ϵ_1 towards r_1;

29   **else if** *I am r_1* **then**

30   **if** *all robots r_p are in T_p, for all $p \geq 3$* **then**

31   **if** $(R_1, R_2)_{\mathcal{D}_p} > \beta_1$ **then**

32   move to T_1;

33   **else**

34   **if** $\alpha_0 < \alpha_i, \forall i \neq 0$ **and** $\alpha_0 < \alpha_1 < \alpha_i, \forall i \neq 0, 1$ **and** $\alpha_0, \alpha_1 < \beta_j, \forall j$ **then**

35   **if** *I am r_p and I am Move Ready* **then**

36   Move to T_p;

choose a positive ϵ independently in the directions of their minimum angle sequences and move the angular distance ϵ towards its neighbor in the direction of its minimum angle sequence. For each nominee, ϵ is chosen independently. So the probability of their choice of the same ϵ is 0. Thus for a different choice of ϵ, one of the angle sequences of a nominee must be minimum and that will be considered as the leader, and leader configuration will be achieved. So a probabilistic algorithm can be given to solve the arbitrary pattern formation problem on the circle for an even number of robots.

7   Conclusion

The arbitrary pattern formation problem is a classical problem in the field of swarm robotics. Till now it has been studied considering the euclidean plane and some discrete domains mostly. In continuous domains, there are certain environments that restrict the movement of the robot in any direction. Any closed curve embedded on a plane is an example of this. In the real world, this kind of environment can be seen everywhere, for example, train lines, road networks, etc. It can be argued that a problem solvable in a continuous circle can be solved on any closed curve. So here, in this paper, this problem has been introduced on a continuous circle for the first time. Here, considering a rotationally asymmetric initial configuration having an odd number of oblivious and silent robots, a distributed deterministic algorithm APF_CIRCLE has been provided that solves the arbitrary pattern formation problem on a continuous circle of fixed radius within $O(n)$ epochs of time under a semi-synchronous scheduler. To justify the initial rotationally asymmetric configuration, it has been also been shown that if the initial configuration is rotationally symmetric then, no deterministic algorithm exists that solves the arbitrary pattern formation problem on a circle. Furthermore, by tweaking the deterministic algorithm a little bit, a probabilistic algorithm has been described that solves arbitrary pattern formation problem for any n number of robots on a circle, where $n \geq 3$.

For the days ahead, it would be really interesting if this problem can be solved under an asynchronous scheduler. Also, another interesting thing to check would be to find out if there is an initial configuration and a target configuration such that for any embedding of the target the time taken by the n robots to form the target is $O(n)$ or not. If this lower bound is $O(n)$ then the algorithm presented here is time optimal otherwise another time the optimal algorithm has to be designed. Furthermore, for a swarm of an even number of robots our algorithm is not deterministic. So, it would be another way to carry forward this research to find out if there is any deterministic algorithm for an even number of robots.

References

1. Bose, K., Adhikary, R., Kundu, M.K., Sau, B.: Arbitrary pattern formation on infinite grid by asynchronous oblivious robots. Theor. Comput. Sci. **815**, 213–227 (2020). https://doi.org/10.1016/j.tcs.2020.02.016, `https://doi.org/10.1016/j.tcs.2020.02.016`

2. Bose, K., Das, A., Sau, B.: Pattern Formation by Robots with Inaccurate Movements. In: Bramas, Q., Gramoli, V., Milani, A. (eds.) 25th International Conference on Principles of Distributed Systems (OPODIS 2021). Leibniz International Proceedings in Informatics (LIPIcs), vol. 217, pp. 10:1–10:20. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany (2022). https://doi.org/10.4230/LIPIcs.OPODIS.2021.10, `https://drops.dagstuhl.de/opus/volltexte/2022/15785`

3. Bose, K., Kundu, M.K., Adhikary, R., Sau, B.: Arbitrary pattern formation by asynchronous opaque robots with lights. Theoretical Computer Science **849**, 138–158 (2021). https://doi.org/https://doi.org/10.1016/j.tcs.2020.10.015, `https://www.sciencedirect.com/science/article/pii/S0304397520305867`

4. Bramas, Q., Tixeuil, S.: Probabilistic asynchronous arbitrary pattern formation (short paper). In: Bonakdarpour, B., Petit, F. (eds.) Stabilization, Safety, and Security of Distributed Systems. pp. 88–93. Springer International Publishing, Cham (2016)

5. Bramas, Q., Tixeuil, S.: Arbitrary pattern formation with four robots. In: Izumi, T., Kuznetsov, P. (eds.) Stabilization, Safety, and Security of Distributed Systems. pp. 333–348. Springer International Publishing, Cham (2018)

6. Cicerone, S., Di Fonso, A., Di Stefano, G., Navarra, A.: Arbitrary pattern formation on infinite regular tessellation graphs. Theoretical Computer Science **942**, 1–20 (2023). https://doi.org/https://doi.org/10.1016/j.tcs.2022.11.021, `https://www.sciencedirect.com/science/article/pii/S0304397522006843`

7. Cicerone, S., Di Stefano, G., Navarra, A.: Embedded pattern formation by asynchronous robots without chirality. Distributed Computing **32**, 291–315 (2019)

8. Cicerone, S., Fonso, A.D., Stefano, G.D., Navarra, A.: Arbitrary pattern formation on infinite regular tessellation graphs. CoRR **abs/2010.14152** (2020), `https://arxiv.org/abs/2010.14152`

9. Czyzowicz, J., Gasieniec, L., Kosowski, A., Kranakis, E.: Boundary patrolling by mobile agents with distinct maximal speeds. In: Demetrescu, C., Halldórsson, M.M. (eds.) Algorithms - ESA 2011 - 19th Annual European Symposium, Saarbrücken, Germany, September 5-9, 2011. Proceedings. Lecture Notes in Computer Science, vol. 6942, pp. 701–712. Springer (2011). https://doi.org/10.1007/978-3-642-23719-5_59, `https://doi.org/10.1007/978-3-642-23719-5_59`

10. Czyzowicz, J., Kranakis, E., Pajak, D., Taleb, N.: Patrolling by robots equipped with visibility. In: Halldórsson, M.M. (ed.) Structural Information and Communication Complexity - 21st International Colloquium, SIROCCO 2014, Takayama, Japan, July 23-25, 2014. Proceedings. Lecture Notes in Computer Science, vol. 8576, pp. 224–234. Springer (2014). https://doi.org/10.1007/978-3-319-09620-9_18, `https://doi.org/10.1007/978-3-319-09620-9_18`

11. Dieudonné, Y., Petit, F., Villain, V.: Leader election problem versus pattern formation problem. In: Lynch, N.A., Shvartsman, A.A. (eds.) Distributed Computing. pp. 267–281. Springer Berlin Heidelberg, Berlin, Heidelberg (2010)

12. Feinerman, O., Korman, A., Kutten, S., Rodeh, Y.: Fast rendezvous on a cycle by agents with different speeds. Theor. Comput. Sci. **688**, 77–85 (2017). https://doi.org/10.1016/j.tcs.2015.12.035, `https://doi.org/10.1016/j.tcs.2015.12.035`

13. Flocchini, P., Killick, R., Kranakis, E., Santoro, N., Yamashita, M.: Gathering and election by mobile robots in a continuous cycle. In: Lu, P., Zhang, G.

(eds.) 30th International Symposium on Algorithms and Computation, ISAAC 2019, December 8-11, 2019, Shanghai University of Finance and Economics, Shanghai, China. LIPIcs, vol. 149, pp. 8:1–8:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2019). https://doi.org/10.4230/LIPIcs.ISAAC.2019.8, `https://doi.org/10.4230/LIPIcs.ISAAC.2019.8`

14. Flocchini, P., Prencipe, G., Santoro, N., Widmayer, P.: Arbitrary pattern formation by asynchronous, anonymous, oblivious robots. Theoretical Computer Science **407**(1), 412–447 (2008). https://doi.org/https://doi.org/10.1016/j.tcs.2008.07.026, `https://www.sciencedirect.com/science/article/pii/S0304397508005379`

15. Ghosh, S., Sharma, A., Goswami, P., Sau, B.: Asynchronous gathering of robots with finite memory on a circle under limited visibility. CoRR **abs/2302.07600** (2023). https://doi.org/10.48550/arXiv.2302.07600, `https://doi.org/10.48550/arXiv.2302.07600`

16. Kundu, M.K., Goswami, P., Ghosh, S., Sau, B.: Arbitrary pattern formation by opaque fat robots on infinite grid. Int. J. Parallel Emergent Distributed Syst. **37**(5), 542–570 (2022). https://doi.org/10.1080/17445760.2022.2088750, `https://doi.org/10.1080/17445760.2022.2088750`

17. Luna, G.A.D., Uehara, R., Viglietta, G., Yamauchi, Y.: Gathering on a circle with limited visibility by anonymous oblivious robots. In: Attiya, H. (ed.) 34th International Symposium on Distributed Computing, DISC 2020, October 12-16, 2020, Virtual Conference. LIPIcs, vol. 179, pp. 12:1–12:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2020). https://doi.org/10.4230/LIPIcs.DISC.2020.12, `https://doi.org/10.4230/LIPIcs.DISC.2020.12`

18. Suzuki, I., Yamashita, M.: Distributed anonymous mobile robots - formation and agreement problems. In: Problems, in the Proceedings of the 3rd International Colloquium on Structural Information and Communication Complexity (SIROCCO '96. pp. 1347–1363 (1996)

19. Yamashita, M., Suzuki, I.: Characterizing geometric patterns formable by oblivious anonymous mobile robots. Theoretical Computer Science **411**(26), 2433–2453 (2010). https://doi.org/https://doi.org/10.1016/j.tcs.2010.01.037, `https://www.sciencedirect.com/science/article/pii/S0304397510000745`